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Unsteady flow of a second-order fluid near a stagnation point

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Two-dimensional flow of a second-order fluid near a stagnation point occurring on a flat plate which is performing harmonic oscillations in its own plane is considered. The equations have been integrated by the Kármán–Pohlhausen method for small values of ω, the frequency of the oscillation of the plate, and the W.B.K. method is applied to solve the equations for high values of ω. The velocity profile within the boundary-layer region and the shearing stress on the plate have been obtained in both the cases. The oscillation of the shearing stress has a phase lead over the oscillation of the plate. This phase lead decreases with increase of the second-order effect for small values of ω.

1. Introduction

Two-dimensional flow of a viscous incompressible fluid near a stagnation point has been discussed by Howarth (1935). The stagnation point occurs at $x = y = 0$, the flow being parallel to the y-axis at infinity and impinging perpendicularly on a flat plate placed along $y = 0$. Rott (1955) has discussed this problem when the plate ($y = 0$) performs harmonic oscillation in its own plane, i.e. in the x-direction, while the flow at $y \to \infty$ remains steady. The flow depends upon a dimensionless number $\kappa = \omega/a$, where ω is the frequency of the oscillation of the plate and a is a constant depending on the flow at infinity. He obtained an exact solution for the case $\kappa \ll 1$ and solved the equations for the case $\kappa \gg 1$ by a method suggested by Wentzel (1926), Brillouin (1926) and Kramers (1926) which is known as the W.B.K. method. In both the cases the oscillation of the shearing stress on the plate has a phase lead over the oscillation of the plate.

The constitutive equation of an incompressible second-order fluid has been given by Coleman & Noll (1960) as

$$\tau_{ij} = -p\delta_{ij} + 2\mu_1 e_{ij}^{(1)} + 2\mu_2 e_{ij}^{(2)} + 4\mu_3 e_{i\alpha}^{(1)} e_{\alpha j}^{(1)}, \qquad (1)$$

where

$$e_{ij}^{(1)} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \qquad (2)$$

$$e_{ij}^{(2)} = \tfrac{1}{2}(a_{i,j} + a_{j,i} + 2v_{m,i}v_{m,j}), \qquad (3)$$

τ_{ij} is the stress tensor, v_i and a_i are the velocity and acceleration vectors respectively, μ_1, μ_2, μ_3 are material constants and p is an indeterminate hydrostatic pressure. The tensors $e_{ij}^{(1)}$ and $e_{ij}^{(2)}$ are known as the first and second rate-of-strain tensors. Solutions of poly-iso-butylene in cetane behave as second-order fluid and these material constants have been determined experimentally by

Markovitz & Brown (see Truesdell 1964). For a 5·4 % solution of poly-isobutylene in cetane it is found that $\mu = 18{\cdot}5$ poises, $\mu_2 = -0{\cdot}2\,\mathrm{g/cm}$ and $\mu_3 = 1{\cdot}0\,\mathrm{g/cm}$. Rajeshwari & Rathna (1962) have solved Howarth's above-mentioned problem for this fluid. Using their results Sharma (1964) has solved the flow near a stagnation point when the main stream outside the boundary layer oscillates in magnitude but not in direction, i.e. the problem solved by Lighthill (1954) for the Newtonian fluid (which can be called a first-order fluid). In this paper the problem discussed by Rott (1955) is solved for a second-order fluid. The equations have been integrated by the Kármán–Pohlhausen method for $\kappa \ll 1$ and by the W.B.K. method for $\kappa \gg 1$. The velocity distribution within the boundary-layer region and the shearing stress on the plate are obtained in both the cases. The results obtained in this paper can be directly applied to the case when the dividing streamline of the oncoming stream oscillates in position and the plate is at rest by superposing a uniform (though not a constant) transverse velocity.

It has been found by the author in an unpublished paper that equation (8) of this paper remains unchanged and the variation of the pressure in the direction of y can be neglected even if the x-axis is taken along any plane curve and the y-co-ordinate as the distance from this curve. Hence using the results obtained in this paper and those obtained by Sharma (1964), and applying the arguments given by Glauert (1956), the following two-dimensional oscillatory motions can be discussed in second-order fluids:

(i) a cylinder of arbitrary cross-section is fixed in the fluid and the stream oscillates in magnitude;

(ii) the cylinder is fixed and the stream oscillates in direction;

(iii) the stream is constant and the cylinder oscillates in the stream direction;

(iv) the stream is constant and the cylinder oscillates in the transverse direction;

(v) the stream is constant and the cylinder oscillates about its axis.

Following Glauert (1956), a quantitative estimate of the torque on a circular cylinder making small transverse oscillations in a constant stream of second-order fluid can be made by using the results obtained in this paper. This torque is a function of μ_1 and μ_2. The value of the constant μ_1 can be calculated by any steady-state viscometer. By measuring the torque experienced by the cylinder in such a motion, experimentally, it may be possible to determine the constant μ_2 for the material.

2. Equations of motion

Consider a plane potential flow parallel to the y-axis at infinity impinging on a flat plate ($y = 0$) which is performing harmonic oscillations of the type $b\cos\omega t$. The velocity components for the potential flow at infinity in the directions of x and y are respectively

$$U = ax, \quad V = -ay. \tag{4}$$

Denoting by u and v the velocity components in the directions of x and y respectively at any point of the fluid, we assume

$$u = axf'(\eta) + b\,\mathrm{Re}[e^{i\omega t}g(\eta)], \quad v = -(a\nu_1)^{\frac{1}{2}}f(\eta), \tag{5}$$

where $\eta = y(a/\nu_1)^{\frac{1}{2}}$, $\nu_1 = \mu_1/\rho$, ρ is the density of the fluid and a prime denotes differentiation with respect to η. f is a real function, but g is complex. The boundary conditions of this problem in terms of $f(\eta)$ and $g(\eta)$ are

$$\left.\begin{aligned} f(\eta) = f'(\eta) = 0, \quad g(\eta) = 1 \quad \text{at} \quad \eta = 0,\\ f'(\eta) \to 1, \quad g(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{6}$$

Further, to ensure that the solution passes over smoothly to that of the potential flow at infinity, we assume

$$\left.\begin{aligned} &f''(\eta) \to 0, \quad f'''(\eta) \to 0, \quad f^{\text{iv}}(\eta) \to 0,\\ &g'(\eta) \to 0, \quad g''(\eta) \to 0, \quad g'''(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{7}$$

Two-dimensional boundary-layer equations for the fluid governed by the constitutive equation (1) have been derived by the author (unpublished) by taking ν_1, ν_2 and ν_3 of the order δ^2 (δ being the boundary-layer thickness) as

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu_1\frac{\partial^2 u}{\partial y^2} + \nu_2\bigg(&\frac{\partial^3 u}{\partial t\,\partial y^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\\ &+ u\frac{\partial^3 u}{\partial x\,\partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\bigg), \end{aligned} \tag{8}$$

$$\frac{\partial}{\partial y}\left[\frac{p}{\rho} - (2\nu_2 + \nu_3)\left(\frac{\partial u}{\partial y}\right)^2\right] \sim O(\delta), \tag{9}$$

where $\nu_2 = \mu_2/\rho$, $\nu_3 = \mu_3/\rho$. Substituting the velocity components from (4) and (5) in equation (8) and equating the coefficient of x and the coefficient of $e^{i\omega t}$ on both sides of the equation thus obtained, we have

$$i\kappa g + gf' - g'f - g'' - \alpha(i\kappa g'' + gf''' - g'f'' + g''f' - g'''f) = 0, \tag{10}$$

$$f'^2 - ff'' - f''' - 1 - \alpha(2f'f''' - f''^2 - ff^{\text{iv}}) = 0, \tag{11}$$

where $\alpha = a\mu_2/\mu_1$. Equation (11) has been solved by Rajeshwari & Rathna (1962) for positive as well as negative values of μ_2. They have integrated the equations by the Kármán–Pohlhausen method by assuming the following form of $f(\zeta)$;

$$\begin{aligned} f(\zeta) = 0{\cdot}125(10\delta_1 - 3a_3)\,\zeta^2 + a_3\,\zeta^3 - 0{\cdot}125(10\delta_1 + 9a_3)\,\zeta^4\\ + 0{\cdot}200(3a_3 + 5\delta_1)\,\zeta^5 - 0{\cdot}125(a_3 + 2\delta_1)\,\zeta^6, \end{aligned}$$

where $\zeta = \eta/\delta_1$, $\delta_1 = \delta(a/\nu_1)$, δ being the boundary-layer thickness. This form of $f(\zeta)$ satisfies all the boundary conditions (6) and (7) and the constants a_3, δ_1 have been determined from the momentum integral equation and from the condition that equation (11) is satisfied at $\zeta = 0$. The values of a_3 and δ_1 are given in table 1 for $\alpha = 0$, $-0{\cdot}1$, $-0{\cdot}2$. Markovitz & Coleman (1964) have shown from thermodynamic considerations that μ_2 should be negative. Markovitz and Brown have determined experimentally this material constant for solutions of poly-iso-butylene in cetane of various concentration and found it to be negative (see Truesdell 1964). Hence in this paper we shall only consider negative values of μ_2.

3. The low-frequency case

When the frequency of the oscillation of the plate is small compared with a, i.e. when $\kappa \ll 1$, we expand $g(\eta)$ in powers of $i\kappa$ as

$$g(\eta) = g_0(\eta) + i\kappa g_1(\eta) + (i\kappa)^2 g_2(\eta) + \ldots, \tag{12}$$

substituting (12) in (10) and equating the coefficient of $i\kappa$ and terms independent of it to zero, we get

$$g_0 f' - g_0' f - g_0'' - \alpha(g_0 f''' - g_0' f'' + g_0'' f' - g_0''' f) = 0, \tag{13}$$

$$g_0 + g_1 f' - g_1' f - g_1'' - \alpha(g_0'' + g_1 f''' - g_1' f'' + g_1'' f' - g_1''' f) = 0. \tag{14}$$

Differentiating (11) with respect to η yields

$$f'f'' - ff''' - f^{\text{iv}} - \alpha(f'f^{\text{iv}} - ff^{\text{v}}) = 0,$$

which suggests that

$$g_0(\eta) = f''(\eta)/f''(0). \tag{15}$$

The function $g_0(\eta)$ has been plotted against η for $\alpha = 0,\ -0{\cdot}1,\ -0{\cdot}2$ in figure 1. The function $g_0(\eta)$ decreases with increase of $(-\alpha)$ for any particular value of η. Changing the variable from η to ζ, (14) reads as

$$\delta_1^3 g_0 + \delta_1^2(g_1 f' - g' f) - \delta_1 g_1'' - \alpha(\delta_1 g_0'' + g_1 f''' - g_1' f'' + g_1'' f' - g_1''' f) = 0, \tag{16}$$

where now a prime denotes differentiation with respect to the new variable ζ. Integrating (16) with respect to ζ from 0 to 1 and using (15), we have

$$\int_0^1 (2\delta_1^2 f' - 4\alpha f''')\, g_1\, d\zeta + \delta_1 g_1'(0) + \frac{1}{f''(0)}\{\delta_1^4 + \alpha\delta_1 f'''(0)\} = 0. \tag{17}$$

Supposing that the boundary conditions satisfied by $g_1(\eta)$ as $\eta \to \infty$ are satisfied at the edge of the boundary layer, we write

$$\left.\begin{aligned} &g_1(\zeta) = 0 \quad \text{at} \quad \zeta = 0;\\ &g_1(\zeta) = g_1'(\zeta) = g_1''(\zeta) = g_1'''(\zeta) = 0 \quad \text{at} \quad \zeta = 1.\end{aligned}\right\} \tag{18}$$

Equation (16) gives

$$\delta_1^3 - \delta g_1''(0) - \alpha\{\delta_1 f^{\text{iv}}(0)/f''(0) - g_1'(0) f''(0)\} = 0. \tag{19}$$

We assume that

$$g_1(\zeta) = (1-\zeta)^4 (A\zeta + B\zeta^2). \tag{20}$$

The above form of $g_1(\zeta)$ satisfies all the conditions of (18) and satisfies (17) and (19) if

$$\delta_1^3 + (8A - 2B)\,\delta_1 + \alpha\{12(10\delta_1 + 9a_3)/(10\delta_1 - 3a_3) + \tfrac{1}{4}(10\delta_1 - 3a_3)\,A\} = 0, \tag{21}$$

$$\begin{aligned} &A\{2\delta_1^2(0{\cdot}001151 a_3 - 0{\cdot}019200\delta_1) + \alpha(0{\cdot}185339 a_3 - 0{\cdot}481401\delta_1)\}\\ &\quad + B\{2\delta_1^2(0{\cdot}000218 a_3 - 0{\cdot}006747\delta_1) + \alpha(0{\cdot}024110 a_3 - 0{\cdot}142856\delta_1)\}\\ &\qquad = -8A - 4(\delta_1^4 - 6\alpha\delta_1 a_3)/(10\delta - 3a_3). \end{aligned} \tag{22}$$

Knowing δ_1 and a_3 from the solution of the steady case equations (21) and (22) give A and B. The values of A and B are given in table 1 for $\alpha = 0,\ -0{\cdot}1,\ -0{\cdot}2$ and the function $g_1(\eta)$ has been plotted against η for these values of α in figure 1.

The maximum value of $g_1(\eta)$ increases with increase of $(-\alpha)$. The graph of both functions $g_0(\eta)$ and $g_1(\eta)$ for $\alpha = 0$ agrees closely with the corresponding ones of the exact solution obtained by Rott (1955). The unsteady part of the shearing stress on the plate is given by

$$\mathrm{Re}[\rho(a\nu_1)^{\frac{1}{2}}\, b\, e^{i\omega t}\{g'(0)\,(1+i\alpha\kappa)+\alpha f''(0)\}], \tag{23}$$

α	δ_1	a_3	A	B	$\tan\phi$
0	3·0332	−4·6051	−1·6364	−1·9456	0·6537κ
−0·1	2·5144	−2·7022	−1·5160	−2·6582	0·5783κ
−0·2	2·0012	−2·0448	−1·4810	−3·4749	0·4370κ

TABLE 1

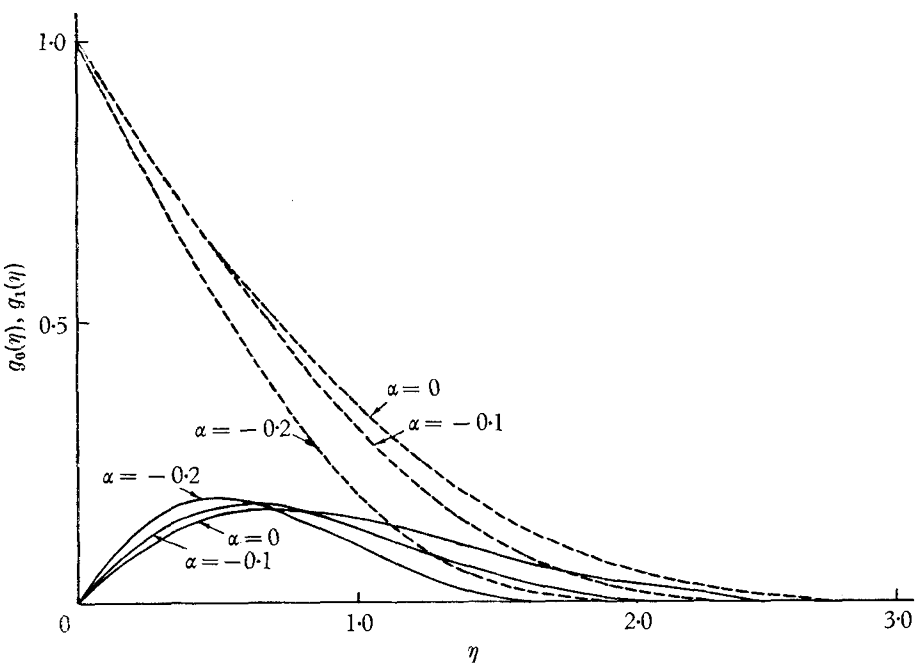

FIGURE 1. First two terms of an approximation to the unsteady velocity profile, as functions of η. – – –, $g_0(\eta)$; ——, $g_1(\eta)$.

which shows that the oscillation of the shearing stress has a phase lead ϕ over the oscillation of the plate given by

$$\phi = \tan^{-1}[\kappa\{g_1'(0)+\alpha g_0'(0)\}/\{g_0'(0)+\alpha f''(0)\}]. \tag{24}$$

This phase lead has been calculated for $\alpha = 0,\ -0·1,\ -0·2$ and is given in table 1. The second-order effects reduce the phase lead.

4. The high-frequency case

When the frequency of the oscillation of the plate is high compared with the value of a, i.e. $\kappa \gg 1$, the W.B.K. method is appropriate. In equation (10) we put

$$g(\eta) = \exp\left(\int_0^\eta s(\eta)\,d\eta\right) \tag{25}$$

and then $s(\eta)$ has to satisfy

$$i\kappa + f' - fs - s^2 - s' - \alpha\{i\kappa(s^2+s') + f''' - sf'' + (s^2+s')f' - f(s''+3ss'+s^3)\} = 0. \tag{26}$$

The value of μ_2/μ_1 is small and negative for most of the fluids which behave as second-order fluids, say it is $-0{\cdot}02667$ for a 6·8 % solution of poly-iso-butylene in cetane (see Markovitz & Coleman 1964). Taking α to be of the order of $1/\kappa$, we put $\alpha\kappa = -m$ in equation (26) and assume

$$s(\eta) = s_0(\eta)\,k^{\frac{1}{2}} + s_1(\eta) + s_2(\eta)\,\kappa^{-\frac{1}{2}} + s_3(\eta)\,\kappa^{-1} + \dots. \tag{27}$$

Putting (27) in (26) and equating like powers of κ, we get

$$i - (1-im)\,s_0^2 = 0, \tag{28}$$

$$2s_1(1-im) + f(1+ms_0^2) = 0, \tag{29}$$

$$f'(1+ms_0^2) - fs_1(1-3ms_0^2) - (1-im)\,(s_1^2 + 2s_0\,s_1 + s_1') = 0, \tag{30}$$

$$fs_2 + (1-im)\,(2s_0\,s_3 + 2s_1\,s_2 + s_2') - ms_0\{-f'' + 2s_1 f' + 3f(s_1' + 3s_1^2)\} = 0. \tag{31}$$

From (28) it is clear that $s_0(\eta)$ is constant, hence $s_0'(\eta)$ has been omitted in writing the equations (29)–(31).

The solution of the equations (28)–(31) is

$$s_0(\eta) = (1+m^2)^{-\frac{1}{4}}\{\cos(\theta+\tfrac{1}{4}\pi) + i\sin(\theta+\tfrac{1}{4}\pi)\}, \tag{32}$$

$$s_1(\eta) = -\{2(1+m^2)\}^{-1}(\cos 4\theta + i\sin 4\theta)f'(\eta), \tag{33}$$

$$\begin{aligned} s_2(\eta) = {} & 0{\cdot}75(1+m^2)^{-\frac{3}{4}}\{\cos(3\theta-\tfrac{1}{4}\pi) + i\sin(3\theta-\tfrac{1}{4}\pi)\}f'(\eta) \\ & + 0{\cdot}125(1+m^2)^{-\frac{7}{4}}[\{\cos(7\theta-\tfrac{1}{4}\pi) + 8m\sin(7\theta-\tfrac{1}{4}\pi)\} \\ & + i\{\sin(7\theta-\tfrac{1}{4}\pi) - 8m\cos(7\theta-\tfrac{1}{4}\pi)\}]f^2(\eta), \end{aligned} \tag{34}$$

where $\theta = \frac{1}{2}\tan^{-1}m$. The expression for $s_3(\eta)$ is complicated but the coefficient of $f''(\eta)$ in the expression is given by

$$0{\cdot}125(1+m^2)\{-11m + i(3-8m^2)\}.$$

The unsteady part of the shearing stress on the plate in terms of $s(\eta)$ and m is given by

$$\begin{aligned} & \rho(a\nu_1)^{\frac{1}{2}}\,b\,e^{i\omega t}[s(0)\,(1-im) - (m/\kappa)f''(0)] \\ & \quad = \rho(a\nu_1)^{\frac{1}{2}}\,b\,e^{i\omega t}[(1+m^2)^{-\frac{1}{4}}\{\cos(\theta+\tfrac{1}{4}\pi) + i\sin(\theta+\tfrac{1}{4}\pi)\}\,\kappa^{\frac{1}{2}} \\ & \qquad - (2m + 0{\cdot}375i)\,\kappa^{-1}f''(0)]. \end{aligned} \tag{35}$$

Hence in this case the phase lead ϕ of the oscillation of the shearing stress on the plate over the oscillation of the plate is given by

$$\tan\phi = \frac{(1+m^2)^{-\frac{1}{4}}\cos(\theta+\frac{1}{4}\pi) - 2mf''(0)\,\kappa^{-\frac{3}{2}}}{(1+m^2)^{-\frac{1}{4}}\sin(\theta+\frac{1}{4}\pi) - 0{\cdot}375f''(0)\,\kappa^{-\frac{3}{2}}}. \tag{36}$$

In this case no comment can be made about the relation of tan ϕ and m, for it involves powers of $\kappa^{\frac{1}{2}}$. All the expressions reduce to the corresponding expressions of Rott for $\alpha = 0$.

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